

# TOPOLOGY OF MIXED HYPERSURFACES OF CYCLIC TYPE

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ABSTRACT. Let  $f_{II}(\mathbf{z}, \bar{\mathbf{z}}) = z_1^{a_1+b_1} \bar{z}_1^{b_1} z_2 + \cdots + z_{n-1}^{a_{n-1}+b_{n-1}} \bar{z}_{n-1}^{b_{n-1}} z_n + z_n^{a_n+b_n} \bar{z}_n z_1$  be a mixed weighted homogeneous polynomial of cyclic type and  $g_{II}(\mathbf{z}) = z_1^{a_1} z_2 + \cdots + z_{n-1}^{a_{n-1}} z_n + z_n^{a_n} z_1$  be the associated weighted homogeneous polynomial in the sense of [2] where  $a_j \geq 1$  and  $b_j \geq 0$  for  $j = 1, \dots, n$ . We show that two links  $S_r^{2n-1} \cap f_{II}^{-1}(0)$  and  $S_r^{2n-1} \cap g_{II}^{-1}(0)$  are diffeomorphic and their Milnor fibrations are isomorphic.

## 1. INTRODUCTION

Let  $f(\mathbf{z}, \bar{\mathbf{z}})$  be a mixed polynomial of complex variables  $\mathbf{z} = (z_1, \dots, z_n)$  given as

$$f(\mathbf{z}, \bar{\mathbf{z}}) := \sum_{i=1}^m c_i \mathbf{z}^{\nu_i} \bar{\mathbf{z}}^{\mu_i},$$

where  $c_i \in \mathbb{C}^*$  and  $\mathbf{z}^{\nu_i} = z_1^{\nu_{i,1}} \cdots z_n^{\nu_{i,n}}$  for  $\nu_i = (\nu_{i,1}, \dots, \nu_{i,n})$  (respectively  $\bar{\mathbf{z}}^{\mu_i} = \bar{z}_1^{\mu_{i,1}} \cdots \bar{z}_n^{\mu_{i,n}}$  for  $\mu_i = (\mu_{i,1}, \dots, \mu_{i,n})$ ). Here  $\bar{z}_j$  represents the complex conjugate of  $z_j$ .

A point  $\mathbf{w} \in \mathbb{C}^n$  is called a *mixed singular point* of  $f(\mathbf{z}, \bar{\mathbf{z}})$  if the gradient vectors of  $\Re f$  and  $\Im f$  are linearly dependent at  $\mathbf{w}$ . Certain restricted classes of mixed polynomials of the variables  $\mathbf{z}$  which admit Milnor fibrations had been considered by J. Seade, see for instance [7, 8]. The last author introduced the notion of the Newton boundary and the concept of non-degeneracy for a mixed polynomial and he showed the existence of Milnor fibration for the class of strongly non-degenerate mixed polynomials [3].

We consider the classes of mixed polynomials which was first introduced by Ruas-Seade-Verjovsky [6] and J. L. Cisneros-Molina [1]. Let  $p_1, \dots, p_n$  and  $q_1, \dots, q_n$  be integers such that  $\gcd(p_1, \dots, p_n) = \gcd(q_1, \dots, q_n) = 1$ . We define the  $S^1$ -action and the  $\mathbb{R}^*$ -action on  $\mathbb{C}^n$  as follows:

$$\begin{aligned} s \circ \mathbf{z} &= (s^{p_1} z_1, \dots, s^{p_n} z_n), \quad s \in S^1, \\ r \circ \mathbf{z} &= (r^{q_1} z_1, \dots, r^{q_n} z_n), \quad r \in \mathbb{R}^*. \end{aligned}$$

If there exists a positive integer  $d_p$  such that  $f(\mathbf{z}, \bar{\mathbf{z}})$  satisfies

$$f(s^{p_1} z_1, \dots, s^{p_n} z_n, \bar{s}^{p_1} \bar{z}_1, \dots, \bar{s}^{p_1} \bar{z}_n) = s^{d_p} f(\mathbf{z}, \bar{\mathbf{z}}), \quad s \in S^1,$$

we say that  $f(\mathbf{z}, \bar{\mathbf{z}})$  is a *polar weighted homogeneous polynomial*. Similarly  $f(\mathbf{z}, \bar{\mathbf{z}})$  is called a *radial weighted homogeneous polynomial* if there exists a positive integer  $d_r$  such that

$$f(r^{q_1} z_1, \dots, r^{q_n} z_n, r^{q_1} \bar{z}_1, \dots, r^{q_n} \bar{z}_n) = r^{d_r} f(\mathbf{z}, \bar{\mathbf{z}}), \quad r \in \mathbb{R}^*.$$

Let  $f$  be a polar and radial weighted homogeneous polynomial. Then  $f$  admits the global Milnor fibration  $f : \mathbb{C}^n \setminus f^{-1}(0) \rightarrow \mathbb{C}^*$ , see for instance [6, 1, 2, 3].

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Let  $f(\mathbf{z}, \bar{\mathbf{z}}) = \sum_{i=1}^m c_i \mathbf{z}^{\nu_i} \bar{\mathbf{z}}^{\mu_i}$  be a mixed polynomial with  $c_j \neq 0, j = 1, \dots, m$ . Put

$$g(\mathbf{z}) := \sum_{i=1}^m c_i \mathbf{z}^{\nu_i - \mu_i}.$$

We call  $g$  the *associated Laurent polynomial* of  $f$ . A mixed polynomial  $f$  is called *simplicial* if  $m \leq n$  and the ranks of the matrices  $N \pm M$  are  $m$  where  $N = (\nu_1, \dots, \nu_n)$  and  $M = (\mu_1, \dots, \mu_n)$ . Here  $\nu_i$  and  $\mu_i$  are considered as column vectors  $\nu_i = {}^t(\nu_{i1}, \dots, \nu_{in})$ ,  $\mu_i = {}^t(\mu_{i1}, \dots, \mu_{in})$ .  $f$  is called *full* if  $m = n$ . A full simplicial mixed polynomial  $f$  and its associated Laurent polynomial  $g$  admit a unique polar weight and a unique radial weight in the above sense [2]. It is useful to consider a graph  $\Gamma$  associated to  $f$ . First we associate a vertex  $v_i$  if  $z_i$  or  $\bar{z}_i$  appears in  $f$ . We join  $v_i$  and  $v_j$  by an edge if there is a monomial  $\mathbf{z}^{\nu_k} \bar{\mathbf{z}}^{\mu_k}$  which contains both variables  $z_i, z_j$ . That is  $\nu_{k,a} + \mu_{k,a} > 0$  for  $a = i, j$ . Most important graphs are a bamboo graph

$$B_n : \overset{v_1}{\bullet} \text{---} \overset{v_2}{\bullet} \text{---} \dots \text{---} \overset{v_{n-1}}{\bullet} \text{---} \overset{v_n}{\bullet}$$

and a cyclic graph  $C_n$  which is obtained from  $B_n$  adding an edge between  $v_n$  and  $v_1$ .

We restrict the Milnor fibrations defined by  $f$  and  $g$  on the complex torus  $\mathbb{C}^{*n}$  where  $\mathbb{C}^{*n} = (\mathbb{C}^*)^n$ . In [2, Theorem 10], it is shown that there exists a canonical diffeomorphism  $\varphi : \mathbb{C}^{*n} \rightarrow \mathbb{C}^{*n}$  which gives an isomorphism of the Milnor fibrations defined by  $f$  and  $g$ :

$$\begin{array}{ccc} \mathbb{C}^{*n} \setminus f^{-1}(0) & \xrightarrow{\varphi} & \mathbb{C}^{*n} \setminus g^{-1}(0) \\ \downarrow f & & \downarrow g \\ \mathbb{C}^* & = & \mathbb{C}^* \end{array}.$$

However the canonical diffeomorphism  $\varphi$  does not extend to  $\mathbb{C}^n \setminus \{O\}$  in general. Here  $O$  is the origin of  $\mathbb{C}^n$ . The exceptional case is a mixed Brieskorn polynomial, for which this canonical diffeomorphism extends as a continuous homeomorphism [6]. In [4], the last author studied the following simplicial polar weighted homogeneous polynomials:

$$\begin{cases} f_{\mathbf{a}, \mathbf{b}}(\mathbf{z}, \bar{\mathbf{z}}) &= z_1^{a_1+b_1} \bar{z}_1^{b_1} + \dots + z_{n-1}^{a_{n-1}+b_{n-1}} \bar{z}_{n-1}^{b_{n-1}} + z_n^{a_n+b_n} \bar{z}_n^{b_n} \\ f_I(\mathbf{z}, \bar{\mathbf{z}}) &= z_1^{a_1+b_1} \bar{z}_1^{b_1} z_2 + \dots + z_{n-1}^{a_{n-1}+b_{n-1}} \bar{z}_{n-1}^{b_{n-1}} z_n + z_n^{a_n+b_n} \bar{z}_n^{b_n} \\ f_{II}(\mathbf{z}, \bar{\mathbf{z}}) &= z_1^{a_1+b_1} \bar{z}_1^{b_1} z_2 + \dots + z_{n-1}^{a_{n-1}+b_{n-1}} \bar{z}_{n-1}^{b_{n-1}} z_n + z_n^{a_n+b_n} \bar{z}_n^{b_n} z_1, \end{cases}$$

where  $a_j \geq 1$  and  $b_j \geq 0$  for  $j = 1, \dots, n$ . Here the notation is the same as in [4]. Note that the graph of  $f_I$  is a bamboo and that of  $f_{II}$  is a cyclic graph. The graph of  $f_{\mathbf{a}, \mathbf{b}}$  is  $n$  disjoint vertices without any edges. A polar weighted homogeneous polynomial  $f_{\mathbf{a}, \mathbf{b}}(\mathbf{z}, \bar{\mathbf{z}})$  and a weighted homogeneous polynomial  $g_{\mathbf{a}}(\mathbf{z})$  are called a *mixed Brieskorn polynomial* and a *Brieskorn polynomial* respectively. A mixed polynomial  $f_I(\mathbf{z}, \bar{\mathbf{z}})$  is called a *simplicial mixed polynomial of bamboo type* and  $f_{II}(\mathbf{z}, \bar{\mathbf{z}})$  is called a *simplicial mixed polynomial of cyclic type* respectively. He showed that two links of  $f_\iota$  and the associated polynomial  $g_\iota(\mathbf{z})$  in a small sphere are isotopic and their Milnor fibrations are isomorphic for  $\iota = (\mathbf{a}, \mathbf{b})$  and  $I$ . He conjectured the assertion will be also true for the case  $f_{II}$ .

## 2. STATEMENT OF THE RESULT

The purpose of this paper is to give a positive answer to the above conjecture. Thus we study the following simplicial polynomial  $f_{II}(\mathbf{z}, \bar{\mathbf{z}})$  and its associated weighted homogeneous polynomial  $g_{II}(\mathbf{z})$ :

$$\begin{cases} f_{II}(\mathbf{z}, \bar{\mathbf{z}}) &= z_1^{a_1+b_1} \bar{z}_1^{b_1} z_2 + \cdots + z_{n-1}^{a_{n-1}+b_{n-1}} \bar{z}_{n-1}^{b_{n-1}} z_n + z_n^{a_n+b_n} \bar{z}_n^{b_n} z_1, \\ g_{II}(\mathbf{z}) &= z_1^{a_1} z_2 + \cdots + z_{n-1}^{a_{n-1}} z_n + z_n^{a_n} z_1, \end{cases}$$

where  $a_j \geq 1$  and  $b_j \geq 0$  for  $j = 1, \dots, n$ . We assume that  $f_{II}$  contains a conjugate  $\bar{z}_j$  for some  $j$ . This implies

(a) there exists  $j \in \{1, \dots, n\}$  such that  $b_j \geq 1$ .

We also assume that  $f_{II}$  is simplicial. As the determinant of  $N - M$  is given by  $a_1 \cdots a_n + (-1)^{n+1}$ , we assume also that

(b) there exists  $k \in \{1, \dots, n\}$  such that  $a_k \geq 2$ .

Though this assumption is not necessary if  $n$  is odd, we assume (b) anyway. Since  $f_{II}(\mathbf{z}, \bar{\mathbf{z}})$  is a polar and radial weighted homogeneous,  $f_{II}(\mathbf{z}, \bar{\mathbf{z}})$  admits a global fibration

$$f_{II} : \mathbb{C}^n \setminus f_{II}^{-1}(0) \rightarrow \mathbb{C}^*$$

[6, 1, 2, 3]. The complex polynomial  $g_{II}(\mathbf{z})$  is a weighted homogeneous polynomial with respect to the same polar weight of  $f_{II}$  and  $g_{II}$  with  $n = 3$  is listed in the classification of weighted homogeneous surfaces in  $\mathbb{C}^3$  with isolated singularity [5]. We consider the hypersurfaces

$$V_f := f_{II}^{-1}(0), \quad V_g := g_{II}^{-1}(0)$$

and respective links

$$K_{f,r} = V_f \cap S_r^{2n-1}, \quad K_{g,r} = V_g \cap S_r^{2n-1}$$

where  $S_r^{2n-1}$  is the  $(2n-1)$ -dimensional sphere centered at the origin  $O$  with radius  $r$ . Then two links  $K_{f,r}$  and  $K_{g,r}$  are smooth for any  $r > 0$  ([2]). We consider the following family of mixed polynomials:

$$\begin{aligned} f_{II,t}(\mathbf{z}, \bar{\mathbf{z}}) &:= (1-t)f_{II}(\mathbf{z}, \bar{\mathbf{z}}) + tg_{II}(\mathbf{z}) \\ &= (1-t)(z_1^{a_1+b_1} \bar{z}_1^{b_1} z_2 + \cdots + z_{n-1}^{a_{n-1}+b_{n-1}} \bar{z}_{n-1}^{b_{n-1}} z_n + z_n^{a_n+b_n} \bar{z}_n^{b_n} z_1) \\ &\quad + t(z_1^{a_1} z_2 + \cdots + z_{n-1}^{a_{n-1}} z_n + z_n^{a_n} z_1) \\ &= \sum_{j=1}^n z_j^{a_j} z_{j+1} \{(1-t)|z_j|^{2b_j} + t\} \end{aligned}$$

where  $0 \leq t \leq 1$ . Here the numbering is modulo  $n$ , so  $z_{n+1} = z_1$ . Though mixed polynomial  $f_{II,t}$  is not radial weighted homogeneous for  $t \neq 0, 1$ ,  $f_{II,t}$  is polar weighted homogeneous for  $0 \leq t \leq 1$  with the same weight  $P = (p_1, \dots, p_n)$  which is characterized by  $a_j p_j + p_{j+1} = d_p$ ,  $j = 1, \dots, n$ . Put

$$V_t = f_{II,t}^{-1}(0), \quad K_{t,r} = S_r^{2n-1} \cap V_t, \quad 0 \leq t \leq 1.$$

Note that

$$\begin{aligned} f_{II,0} &= f_{II}, \quad f_{II,1} = g_{II} \\ V_f &= V_0, \quad K_{f,r} = K_{0,r}, \quad V_g = V_1, \quad K_{g,r} = K_{1,r}. \end{aligned}$$

First recall that  $V_t$  has an isolated mixed singularity at the origin  $O$  and  $V_t \setminus \{O\}$  is non-singular for any  $0 \leq t \leq 1$  by [4, Lemma 9]. Our main result is:

**Transversality Theorem 1.** *Let  $V_t$  be as above. For any fixed  $r > 0$ , the sphere  $S_r^{2n-1}$  and the family of hypersurfaces  $V_t$  are transversal for  $0 \leq t \leq 1$ .*

### 3. PROOF OF TRANSVERSALITY THEOREM 1

**3.1. Strategy of the proof.** We follow the recipe of [4]. First recall that

$$\begin{aligned} f_{II,t}(\mathbf{z}, \bar{\mathbf{z}}) &:= (1-t)f_{II}(\mathbf{z}, \bar{\mathbf{z}}) + tg_{II}(\mathbf{z}) \\ &= \sum_{j=1}^n z_j^{a_j} \bar{z}_{j+1} \{(1-t)|z_j|^{2b_j} + t\}. \end{aligned}$$

Recall that  $V_t$  is non-singular off the origin by [4]. To show the transversality of the sphere  $S_{r_0}^{2n-1}$  and  $V_t$ , we have to show that the Jacobian matrix of  $\Re f_{II,t}$ ,  $\Im f_{II,t}$  and  $\rho(\mathbf{z})$  has rank 3 at every intersection  $\mathbf{w} \in S_{r_0}^{2n-1} \cap V_t$ . Here  $\rho(\mathbf{z}) = \|\mathbf{z}\|^2$ , the square of the radius  $\|\mathbf{z}\|$ . However this computation is extremely complicated. Instead, we follow the recipe of [4]. We will show *the existence of a tangent vector  $\mathbf{v} \in T_{\mathbf{w}}V_t$  which is not tangent to the sphere  $S_{r_0}^{2n-1}$ .*

Take a point  $\mathbf{w} = (w_1, \dots, w_n) \in V_t \cap S_{r_0}^{2n-1}$  and fix it hereafter. To find such a vector  $\mathbf{v}$ , we will construct a real analytic path

$$\mathbf{w}(s) = (r_1(s)w_1, \dots, r_n(s)w_n)$$

on a neighborhood of  $s = 0$  so that  $\mathbf{w}(0) = \mathbf{w}$  and

$$(1) \quad f_{II,t}(\mathbf{w}(s), \bar{\mathbf{w}}(s)) = (s+1)f_{II,t}(\mathbf{w}, \bar{\mathbf{w}})$$

where  $r_j(s)$ ,  $j = 1, \dots, n$  are real-valued functions on  $|s| \ll 1$  which satisfy certain functional equalities. The equality (1) implies that the curve  $\mathbf{w}(s)$  is an embedded curve in  $V_t$  with  $\mathbf{w}(0) = \mathbf{w}$ . Then we define the vector as the tangent vector of this curve at  $s = 0$ :

$$(2) \quad \mathbf{v} = \frac{d\mathbf{w}}{ds}(0).$$

To find such a path  $\mathbf{w}(s) = (r_1(s)w_1, \dots, r_n(s)w_n)$ , we solve a certain functional equation, using the inverse mapping theorem.

**3.2. Construction of  $\mathbf{w}(s)$ .** First we consider the following map:

$$\begin{aligned} \Phi_{\mathbf{w}} : \mathbb{R}^{n+1} &\rightarrow \mathbb{R}^{n+1} \\ (r_1, \dots, r_n, s) &\mapsto (h_1, \dots, h_n, s), \end{aligned}$$

where  $h_j$  is a polynomial function of variables  $r_1, \dots, r_n$  and  $s$  defined by

$$(3) \quad h_j = r_j^{a_j} r_{j+1} \{(1-t)|w_j|^{2b_j} r_j^{2b_j} + t\} - (s+1)\{(1-t)|w_j|^{2b_j} + t\}, \quad j = 1, \dots, n$$

where  $t$  is fixed on  $0 \leq t \leq 1$ . We want to solve the equations  $h_1 = \dots = h_n = 0$  in  $r_1, \dots, r_n$  expressing  $r_j$  as a function of  $s$  so that we get the system of equations

$$(4) \quad h_j(r_1(s), \dots, r_n(s), s) \equiv 0, \quad j = 1, \dots, n.$$

This equality is equivalent to (1) which is more explicitly written as

$$\begin{aligned} f_{II,t}(\mathbf{w}(s), \bar{\mathbf{w}}(s)) &= (s+1)f_{II,t}(\mathbf{w}, \bar{\mathbf{w}}) \quad \text{where} \\ f_{II,t}(\mathbf{w}(s), \bar{\mathbf{w}}(s)) &= \sum_{j=1}^n (r_j(s)w_j)^{a_j} (r_{j+1}(s)w_{j+1}) \{(1-t)|w_j|^{2b_j} r_j(s)^{2b_j} + t\}, \\ (s+1)f_{II,t}(\mathbf{w}, \bar{\mathbf{w}}) &= (s+1) \sum_{j=1}^n w_j^{a_j} w_{j+1} \{(1-t)|w_j|^{2b_j} + t\}. \end{aligned}$$

We will solve the functional equality (1) using the inverse mapping theorem.

**Lemma 1.** *The Jacobian matrix  $J(\Phi_{\mathbf{w}})$  has rank  $n+1$  at  $(r_1, \dots, r_n, s) = (1, \dots, 1, 0)$ ,  $0 < t < 1$  and  $r_0 > 0$  where*

$$J(\Phi_{\mathbf{w}}) = \begin{pmatrix} \frac{\partial h_1}{\partial r_1} & \dots & \frac{\partial h_1}{\partial r_n} & \frac{\partial h_1}{\partial s} \\ & \ddots & & \\ \frac{\partial h_n}{\partial r_1} & \dots & \frac{\partial h_n}{\partial r_n} & \frac{\partial h_n}{\partial s} \\ \frac{\partial s}{\partial r_1} & \dots & \frac{\partial s}{\partial r_n} & \frac{\partial s}{\partial s} \end{pmatrix}.$$

*Proof.* By a direct computation, the Jacobian matrix of  $\Phi_{\mathbf{w}}$  is given as

$$J(\Phi_{\mathbf{w}}) = \begin{pmatrix} \alpha_{1,1} & \alpha_{1,2} & 0 & \dots & 0 & -\beta_1 \\ 0 & \alpha_{2,2} & \alpha_{2,3} & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & \alpha_{n-1,n-1} & \alpha_{n-1,n} & \vdots \\ \alpha_{n,1} & 0 & \dots & 0 & \alpha_{n,n} & -\beta_n \\ 0 & \dots & \dots & \dots & 0 & 1 \end{pmatrix},$$

where

$$\begin{aligned} \alpha_{j,j} &= r_j^{a_j-1} r_{j+1} \{(1-t)|w_j|^{2b_j} (a_j + 2b_j) r_j^{2b_j} + a_j t\}, \\ \alpha_{j,j+1} &= r_j^{a_j} \{(1-t)|w_j|^{2b_j} r_j^{2b_j} + t\}, \\ \beta_j &= \{(1-t)|w_j|^{2b_j} + t\}, \quad j = 1, \dots, n. \end{aligned}$$

Since  $0 < t < 1$ ,  $\alpha_{j,j}$  and  $\alpha_{j,j+1}$  are positive real numbers for each  $j = 1, \dots, n$ . The determinant  $\det J(\Phi_{\mathbf{w}})$  is given as

$$\begin{aligned} (5) \quad \det J(\Phi_{\mathbf{w}}) &= \alpha_{1,1} \dots \alpha_{n,n} + (-1)^{n+1} \alpha_{1,2} \dots \alpha_{n-1,n} \alpha_{n,1} \\ &= \prod_{j=1}^n r_j^{a_j} \{(1-t)|w_j|^{2b_j} (a_j + 2b_j) r_j^{2b_j} + a_j t\} \\ &\quad + (-1)^{n+1} \prod_{j=1}^n r_j^{a_j} \{(1-t)|w_j|^{2b_j} r_j^{2b_j} + t\}. \end{aligned}$$

The proof of Lemma 1 is reduced to the following assertion.

**Assertion 1.**  $\det J(\Phi_{\mathbf{w}}) > 0$ .

*Proof.* (i) If  $n$  is an odd number,  $\det J(\Phi_{\mathbf{w}})$  at  $(1, \dots, 1, 0)$  is obviously positive.  
(ii) Suppose that  $n$  is a positive even number. Consider

$$\alpha'_{j,j} := r_j^{a_j} \{(1-t)|w_j|^{2b_j}(a_j + 2b_j)r_j^{2b_j} + a_j t\}.$$

Note that  $\prod_{j=1}^n \alpha_{j,j} = \prod_{j=1}^n \alpha'_{j,j}$ . We have the following.

$$\begin{aligned} \det J(\Phi_{\mathbf{w}}) &= \prod_{j=1}^n \alpha_{j,j} - \prod_{j=1}^n \alpha_{j,j+1} = \prod_{j=1}^n \alpha'_{j,j} - \prod_{j=1}^n \alpha_{j,j+1} \\ \det J(\Phi_{\mathbf{w}}) \geq 0 &\iff \prod_{j=1}^n \frac{\alpha'_{j,j}}{\alpha_{j,j+1}} \geq 1. \end{aligned}$$

As  $\alpha'_{j,j} \geq \alpha_{j,j+1}$ , the equality takes place if  $\alpha'_{j,j} = \alpha_{j,j+1}$  for  $j = 1, \dots, n$ . We assume that  $\alpha'_{j,j} = \alpha_{j,j+1}$  for any  $j, 1 \leq j \leq n$ . Then

$$(1-t)|w_j|^{2b_j}(a_j + 2b_j) + a_j t = (1-t)|w_j|^{2b_j} + t, \quad j = 1, \dots, n$$

and this is the case if and only if  $(w_j, a_j) = (0, 1)$  or  $(a_j, b_j) = (1, 0)$ . Thus the Jacobian of  $\Phi_{\mathbf{w}}$  at  $(1, \dots, 1, 0)$  is equal to 0 if and only if  $(w_j, a_j) = (0, 1)$  or  $(a_j, b_j) = (1, 0)$  for  $j = 1, \dots, n$ . However this case does not happen, as there exists  $j$  with  $a_j \geq 2$ . Thus the assertion is proved. This completes also the proof of Lemma 1.  $\square$

Now we are ready to prove the transversality of  $S_{r_0}^{2n-1}$  and  $V_t$  for any  $r_0 > 0$  and  $0 \leq t \leq 1$ .

**3.3. Proof of Transversality Theorem.** The assertion is known for  $t = 0, 1$  by [3]. Thus we assume that  $0 < t < 1$ . Recall that  $f_{II,t} : \mathbb{C}^n \rightarrow \mathbb{C}$  has a unique singularity at the origin  $O$  for any  $0 \leq t \leq 1$  by [4, Lemma 9]. As the codimension of  $T_{\mathbf{w}}S_{r_0}^{2n-1}$  in  $\mathbb{C}^n \cong \mathbb{R}^{2n}$  is 1, to show the transversality, it suffices to show the existence of a vector  $\mathbf{v} \in T_{\mathbf{w}}V_t$  with  $\mathbf{v} \notin T_{\mathbf{w}}S_{r_0}^{2n-1}$ .

For a given  $\mathbf{w} = (w_1, \dots, w_n) \in V_t$ , we consider the nullity set  $I_{\mathbf{w}} = \{i \mid w_i = 0\}$ .

**Case 1.**  $I_{\mathbf{w}} = \emptyset$ . This is the most essential case and does not appear for the mixed polynomials  $f_{\mathbf{a},\mathbf{b}}$  and  $f_I$ . The corresponding graph is cyclic.

By Lemma 1, the Jacobian of  $\Phi_{\mathbf{w}}$  at  $(1, \dots, 1, 0)$  is non-zero. By the Inverse mapping theorem, there exist a neighborhood  $U \subset \mathbb{R}^{n+1}$  of  $(1, \dots, 1, 0)$  and a neighborhood  $W \subset \mathbb{R}^{n+1}$  of  $\Phi_{\mathbf{w}}(1, \dots, 1, 0) = (0, \dots, 0)$  and a real analytic mapping  $\Psi_{\mathbf{w}} = (\psi_1, \dots, \psi_n, \text{id}) : W \rightarrow U$  so that

$$\Phi_{\mathbf{w}} \circ \Psi_{\mathbf{w}} = \text{id}_W \quad \text{and} \quad \Psi_{\mathbf{w}} \circ \Phi_{\mathbf{w}} = \text{id}_U.$$

Put  $\mathbf{0} := (0, \dots, 0) \in \mathbb{R}^n$  and consider  $V \subset \mathbb{R} := W \cap (\{\mathbf{0}\} \times \mathbb{R})$ , a neighborhood of  $0 \in \mathbb{R}$  and define smooth functions  $r_j : V \rightarrow \mathbb{R}$  of the variable  $s$  by  $r_j(s) := \psi_j(0, \dots, 0, s)$ . Note that  $r_j(0) = 1$ . We have the equalities:

$$h_j(r_1(s), \dots, r_n(s), s) \equiv 0, \quad s \in V, \quad j = 1, \dots, n.$$

As we have seen in the above discussion, this implies

$$r_j^{a_j}(s)r_{j+1}(s)\{(1-t)|w_j|^{2b_j}r_j(s)^{2b_j} + t\} - (s+1)\{(1-t)|w_j|^{2b_j} + t\} \equiv 0,$$

which implies

$$\begin{aligned} f_{II,t}(\mathbf{w}(s), \bar{\mathbf{w}}(s)) &= \sum_{j=1}^n r_j(s)^{a_j} r_{j+1}(s) w_j^{a_j} w_{j+1} \{(1-t)|w_j|^{2b_j} r_j(s)^{2b_j} + t\} \\ &= (s+1) f_{II,t}(\mathbf{w}, \bar{\mathbf{w}}). \end{aligned}$$

Thus  $f_{II,t}(\mathbf{w}(s), \bar{\mathbf{w}}(s)) \equiv 0$ . Put  $\mathbf{v} = \frac{d\mathbf{w}}{ds}(0)$ . We have  $\mathbf{v} \in T_{\mathbf{w}}V_t$  by the definition. Now to finish the proof of the transversality assertion, we need only to show

**Assertion 2.**  $\mathbf{v} \neq \mathbf{0}$  and  $\mathbf{v} \notin T_{\mathbf{w}}S_{r_0}^{2n-1}$ .

To prove the assertion, we consider the differential in  $s$  of

$$h_j(r_1(s), \dots, r_n(s), s) = r_j(s)^{a_j} r_{j+1}(s) \{(1-t)|w_j|^{2b_j} r_j(s)^{2b_j} + t\} - (s+1) \{(1-t)|w_j|^{2b_j} + t\}.$$

By a direct computation, we get the equality

$$\begin{aligned} \frac{d}{ds} h_j(r_1(s), \dots, r_n(s), s) &= \left( \sum_{k=1}^n \frac{\partial h_j}{\partial r_k} \frac{dr_k}{ds} \right) - \beta_j \\ &= \alpha_{j,j} \frac{dr_j}{ds} + \alpha_{j,j+1} \frac{dr_{j+1}}{ds} - \beta_j \equiv 0 \end{aligned}$$

where

$$\begin{aligned} \alpha_{j,j} &= r_j^{a_j-1} r_{j+1} \{(1-t)|w_j|^{2b_j} (a_j + 2b_j) r_j^{2b_j} + a_j t\}, \\ \alpha_{j,j+1} &= r_j^{a_j} \{(1-t)|w_j|^{2b_j} r_j^{2b_j} + t\}, \\ \beta_j &= \{(1-t)|w_j|^{2b_j} + t\} \end{aligned}$$

for  $j = 1, \dots, n$ . The above equality can be written as

$$\begin{aligned} A \begin{pmatrix} \frac{dr_1}{ds} \\ \vdots \\ \frac{dr_n}{ds} \end{pmatrix} &= \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix} \quad \text{where} \\ A &:= \begin{pmatrix} \alpha_{1,1} & \alpha_{1,2} & 0 & \dots & 0 \\ 0 & \alpha_{2,2} & \alpha_{2,3} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & \alpha_{n-1,n-1} & \alpha_{n-1,n} \\ \alpha_{n,1} & 0 & \dots & 0 & \alpha_{n,n} \end{pmatrix}. \end{aligned}$$

Observe that the above equality says  $\frac{dr_j}{ds}(s)$  is independent of  $s$ . By Lemma 1, the determinant of  $A$  is positive. We first consider the differential  $\frac{dr_1}{ds}$  and will show that  $\frac{dr_1}{ds}(0) \geq 0$ . Put  $m = \lfloor n/2 \rfloor$ , the largest integer such that  $m \leq n/2$ . By the Cramer's formula, the

differential  $\frac{dr_1}{ds}$  of  $r_1$  is equal to

$$\begin{aligned} \frac{dr_1}{ds} &= \frac{1}{\det A} \det \begin{pmatrix} \beta_1 & \alpha_{1,2} & 0 & \dots & 0 \\ \vdots & \alpha_{2,2} & \alpha_{2,3} & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \alpha_{n-1,n} \\ \beta_n & 0 & \dots & 0 & \alpha_{n,n} \end{pmatrix} \\ &= \frac{1}{\det A} \sum_{j=1}^n (-1)^{j-1} A_{j-1} \beta_j A'_{j+1} \\ &= \begin{cases} \frac{1}{\det A} \sum_{k=1}^m (A_{2k-2} \beta_{2k-1} A'_{2k} - A_{2k-1} \beta_{2k} A'_{2k+1}), & n = 2m \\ \frac{1}{\det A} \sum_{k=1}^m (A_{2k-2} \beta_{2k-1} A'_{2k} - A_{2k-1} \beta_{2k} A'_{2k+1}) + A_{n-1} \beta_n A'_{n+1}, & n = 2m + 1 \end{cases} \end{aligned}$$

where

$$A_{j-1} = \begin{cases} 1 & j = 1 \\ \prod_{\ell=1}^{j-1} \alpha_{\ell, \ell+1} & j \geq 2 \end{cases}, \quad A'_{j+1} = \begin{cases} \prod_{\ell=j}^{n-1} \alpha_{\ell+1, \ell+1} & j \leq n-1 \\ 1 & j = n \end{cases}.$$

We have

$$A_{2k-2} \beta_{2k-1} A'_{2k} - A_{2k-1} \beta_{2k} A'_{2k+1} = A_{2k-2} A'_{2k+1} (\beta_{2k-1} \alpha_{2k, 2k} - \alpha_{2k-1, 2k} \beta_{2k}).$$

As  $\alpha_{j, j+1}(1, \dots, 1) = \beta_j$  for  $j = 1, \dots, n$ , we observe that

$$\begin{aligned} & \beta_{2k-1} \alpha_{2k, 2k}(1, \dots, 1) - \alpha_{2k-1, 2k}(1, \dots, 1) \beta_{2k} \\ &= \beta_{2k-1} \{ \alpha_{2k, 2k}(1, \dots, 1) - \beta_{2k} \} \\ &= \beta_{2k-1} \{ (1-t) |w_{2k}|^{2b_{2k}} (a_{2k} + 2b_{2k}) + a_{2k} t - (1-t) |w_{j+1}|^{2b_{j+1}} - t \} \\ &= \beta_{2k-1} \{ (1-t) |w_{2k}|^{2b_{2k}} (a_{2k} + 2b_{2k} - 1) + (a_{2k} - 1)t \} \geq 0. \end{aligned}$$

The equality holds only if  $a_{2k} = 1$  and  $b_{2k} = 0$ . Note that  $w_i \neq 0$  for any  $i = 1, \dots, n$  by the assumption. Anyway we have

$$\frac{dr_1}{ds}(0) \geq 0.$$

If  $n$  is an odd integer, we see that  $\frac{dr_1}{ds}(0) > 0$  by the last unpaired term:  $\frac{dr_1}{ds}(0) \geq A_{n-1} \beta_n A'_{n+1} > 0$ . If there exists some  $k$  such that  $a_{2k} \geq 2$ , we have also the strict inequality:  $\frac{dr_1}{ds}(0) > 0$ .

Next we consider  $\frac{dr_k}{ds}$  for  $k \geq 2$ . First observe that our polynomial  $f_{II, t}$  has a symmetry for the cyclic permutation of the coordinates  $\sigma = (1, 2, \dots, n)$ . Secondly after cyclic change of coordinates, say  $\mathbf{z}' = (z'_1, \dots, z'_n) = (z_{\sigma^i(1)}, \dots, z_{\sigma^i(n)})$ , the equality (3) does not change. That is,  $\mathbf{w}'(s) = (r_{\sigma^i(1)} w_{\sigma^i(1)}, \dots, r_{\sigma^i(n)} w_{\sigma^i(n)})$  is the obtained solution curve. The tangent vector  $\mathbf{v}' = \frac{d\mathbf{w}'}{ds}(0)$  is also equal to  $\mathbf{v}$  after the corresponding cyclic permutation of coordinates. Therefore we can apply the above argument to have the inequality  $\frac{dr_{\sigma^i(1)}}{ds}(0) \geq 0$  for any  $i$ . As we have some  $j$  with  $a_j \geq 2$ , this implies

$$\frac{dr_{j-1}}{ds}(0) > 0.$$



Now we are ready to show that  $\mathbf{v} \neq \mathbf{0}$  and  $\mathbf{v} \notin T_{\mathbf{w}}S_{r_0}^{2n-1}$ . By the assumption of  $\mathbf{w}$ , the path  $\mathbf{w}(s)$  satisfies

$$\begin{aligned} f_{II,t}(\mathbf{w}(s), \bar{\mathbf{w}}(s)) &= (s+1)f_{II,t}(\mathbf{w}, \bar{\mathbf{w}}) \equiv 0, \\ \frac{d\|\mathbf{w}(s)\|^2}{ds}|_{s=0} &= 2 \sum_{j=1}^n r_j(0) \frac{dr_j}{ds}(0) |w_j|^2 = 2 \sum_{j=1}^n \frac{dr_j}{ds}(0) |w_j|^2 > 0. \end{aligned}$$

This implies that  $\mathbf{v} \neq \mathbf{0}$  and  $\mathbf{v} \notin T_{\mathbf{w}}S_{r_0}^{2n-1}$ .

**Case 2.** Now we consider the case  $I_{\mathbf{w}} \neq \emptyset$ . Put  $I_{\mathbf{w}}^c$  be the complement of  $I_{\mathbf{w}}$  and  $\mathbb{C}^{*I_{\mathbf{w}}^c} = \{\mathbf{z} \in \mathbb{C}^n \mid z_i = 0, i \in I_{\mathbf{w}}\}$ . We consider the mixed polynomial  $f'(\mathbf{z}, \bar{\mathbf{z}}) = f_{II,t}|_{\mathbb{C}^{*I_{\mathbf{w}}^c}}$ . Let  $J$  be the set of indices  $j$  for which  $z_j$  or  $\bar{z}_j$  appears in  $f'$ . Note that  $J \subset I_{\mathbf{w}}^c$  but it can be a proper subset.

Case 2-1. Assume that  $f' \equiv 0$ , i.e.,  $J = \emptyset$ . We take simply a real analytic path as follows:

$$\mathbf{w}(s) = (s+1)\mathbf{w}$$

for  $s \in \mathbb{R}$ . Since  $\mathbf{w} \in V_t \setminus \{O\}$ , we observe that

$$f_{II,t}(\mathbf{w}(s), \bar{\mathbf{w}}(s)) \equiv 0, \quad \frac{d\|\mathbf{w}(s)\|^2}{ds}|_{s=0} = 2\|\mathbf{w}\|^2 > 0.$$

Case 2-2. Assume that  $f' \not\equiv 0$ . Then using the connected components of the graph of  $f'$ , we can express  $f'$  uniquely as follows.

$$f'(\mathbf{z}, \bar{\mathbf{z}}) = f_1(\mathbf{z}_{I_1}) + \cdots + f_k(\mathbf{z}_{I_k})$$

where the graph of  $f_i$  is a bamboo and the variables of  $f_i, f_j, i \neq j$  are disjoint and the above expression is a join type expression. Here  $I_i$  be the set of indices of variables of  $f_i$  and  $\mathbf{z}_{I_i} = (z_j)_{j \in I_i}$  are the variables of  $f_i$  for  $i = 1, \dots, k$ . We have the equality  $\cup_{i=1}^k I_i = J$  and  $I_i \cap I_j = \emptyset$  for  $i \neq j$ . Put  $\mathbb{C}^{I_i} = \{\mathbf{z} \in \mathbb{C}^n \mid z_j = 0, j \notin I_i\}$ . Fixing  $i$ , we will construct a curve  $\mathbf{w}_{I_i}(s)$  on  $\mathbb{C}^{*I_i}$  so that

$$f_i(\mathbf{w}_{I_i}(s)) = (s+1)f_i(\mathbf{w}_{I_i}).$$

The construction of the curve  $\mathbf{w}_{I_i}(s)$  can be reduced to the argument of [4, Lemma 10]. We will give briefly the proof which is based on the argument of [4].

For  $j \notin J$ , we put  $w_j(s) = w_j$  and  $\mathbf{w}_{J^c}(s) = \mathbf{w}_{J^c} \in \mathbb{C}^{J^c}$  where  $\mathbb{C}^{J^c} = \{\mathbf{z} \in \mathbb{C}^n \mid z_{j'} = 0, j' \in J\}$ . Here  $\mathbf{w}_{J^c}$  is the projection of  $\mathbf{w}$  to  $\mathbb{C}^{J^c}$ . For each  $i = 1, \dots, k$ , we will construct a curve  $\mathbf{w}_{I_i}(s)$  on  $\mathbb{C}^{I_i}$  and define  $\mathbf{w}_J(s) = \mathbf{w}_{I_1}(s) + \cdots + \mathbf{w}_{I_k}(s)$ . Finally we define a curve  $\mathbf{w}(s) = \mathbf{w}_{J^c} + \mathbf{w}_J(s) \in \mathbb{C}^n$  so that

$$\begin{aligned} f_i(\mathbf{w}_{I_i}(s)) &= (s+1)f_i(\mathbf{w}_{I_i}), \\ f_{II,t}(\mathbf{w}(s)) &= f'(\mathbf{w}(s)) \\ &= f_1(\mathbf{w}_{I_1}(s)) + \cdots + f_k(\mathbf{w}_{I_k}(s)) \\ &= (s+1)\{f_1(\mathbf{w}_{I_1}) + \cdots + f_k(\mathbf{w}_{I_k})\} \\ &= (s+1)f_{II,t}(\mathbf{w}) \equiv 0. \end{aligned}$$

So we fix  $i$ . For simplicity's sake, we assume  $I_i = \{j \mid \nu_i \leq j \leq \mu_i\}$  with  $\mu_i \leq n$ . The last assumption  $\mu_i \leq n$  is for the simplicity of the indices. This implies that

$$f_i(\mathbf{z}_{I_i}) = \sum_{j=\nu_i}^{\mu_i-1} z_j^{a_j} z_{j+1} \{|z_j|^{2b_j}(1-t) + t\}.$$

We will show that there exists a differentiable positive real-valued function solution  $(r_{\nu_i}(s), \dots, r_{\mu_i}(s))$  of the following equation so that  $w_j(s) = r_j(s)w_j$ ,  $j \in I_i$  and

$$w_j^{a_j}(s)w_{j+1}(s)\{|w_j(s)|^{2b_j}(1-t) + t\} = (s+1)w_j^{a_j}w_{j+1}\{|w_j|^{2b_j}(1-t) + t\}$$

for  $j = \nu_i, \dots, \mu_i - 1$ . We first consider the equality

$$(E_j'') : \quad r_j^{a_j}\{|w_j|^{2b_j}r_j^{2b_j}(1-t) + t\} = s_j\{|w_j|^{2b_j}(1-t) + t\}$$

where  $s_j := (s+1)/r_{j+1}$ ,  $\nu_i \leq j \leq \mu_i - 1$ .

First we define  $r_{\mu_i} = 1$  to start with. The left side of  $(E_j'')$  is a monotone increasing function of  $r_j > 0$ . Thus assuming  $s_j > 0$  and considering  $s_j$  as an independent variable, we can solve  $(E_j'')$  in  $r_j$  as a function of  $s_j$ . Thus we put  $r_j = \psi_j(s_j)$ . We claim

**Assertion 3.**

$$(6) \quad \psi_j(1) = 1, \quad \frac{d\psi_j}{ds_j}(s_j) > 0,$$

$$(7) \quad \psi_j(s_j)^{a_j} \leq s_j, \quad j = i_1, \dots, i_{\ell-1}.$$

*Proof.* For  $j = \mu_i - 1$ , the assertion is obvious. Assume that  $j < \mu_i - 1$ . The assertion (6) is obvious. The assertion (7) follows from (6), as  $\psi(s_j)$  is monotone increasing on  $s_j$  and

$$|w_j|^{2b_j}r_j^{2b_j}(1-t) + t \geq |w_j|^{2b_j}(1-t) + t, \quad r_j \geq 1.$$

□

Now we define  $s_j(s)$  and  $r_j(s)$  inductively from  $j = \mu_i$  downward (more precisely from the right end vertex of the graph to the left) as follows:

$$r_{\mu_i}(s) = 1, \quad s_j(s) = (s+1)/r_{j+1}(s), \quad r_j(s) = \psi_j(s_j(s))$$

for  $\nu_i \leq j \leq \mu_i - 1$ .

**Assertion 4.**  $s_j(s) \geq 1$  and  $r_j(s) \geq 1$  for  $j = \nu_i, \dots, \mu_i - 1$  and  $s \geq 0$ .

*Proof.* We show the assertion by a downward induction. For  $j = \mu_i - 1$ , the assertion is obvious. By the inequality (7), we have for  $j < \mu_i - 1$

$$\begin{aligned} s_j(s)^{a_{j+1}} &= \left( \frac{s+1}{r_{j+1}(s)} \right)^{a_{j+1}} = \frac{(s+1)^{a_{j+1}}}{\psi_{j+1}(s_{j+1}(s))^{a_{j+1}}} \\ &\geq \frac{(s+1)^{a_{j+1}}}{s_{j+1}(s)} = (s+1)^{a_{j+1}-1}r_{j+2}(s) \end{aligned}$$

for  $s \geq 0$ . By the definition of  $r_j(s)$  and Assertion 3,  $s_j(s) \geq 1$  and  $r_j(s) \geq 1$  for  $j = \nu_i, \dots, \mu_i - 1$  and  $s \geq 0$ . □

By Assertion 3 and Assertion 4, we see easily that

$$\begin{cases} \frac{dr_{\mu_i-1}}{ds}(0) > 0 \\ \frac{dr_j}{ds}(0) \geq 0, \quad j = \nu_i, \dots, \mu_i - 2. \end{cases}$$

Now we define the curve  $\mathbf{w}_{I_i}(s)$  on  $\mathbb{C}^{I_i}$  by

$$w_j(s) = r_j(s)w_j, \quad j \in I_i.$$

As a vector in  $\mathbb{C}^n$ , the other coefficients of  $\mathbf{w}_{I_i}(s)$  are defined to be zero. Then by the construction we have

$$\begin{aligned}\mathbf{w}_{I_i}(0) &= \mathbf{w}_{I_i}, \quad f_i(\mathbf{w}_{I_i}(s), \bar{\mathbf{w}}_{I_i}(s)) = (s+1)f_i(\mathbf{w}_{I_i}, \bar{\mathbf{w}}_{I_i}), \\ \frac{d\|\mathbf{w}_{I_i}\|^2}{ds}\big|_{s=0} &\geq 2\frac{dr_{\mu_i-1}}{ds}(0)|w_{\mu_i-1}|^2 > 0\end{aligned}$$

where  $|s| \ll 1$  and  $1 \leq i \leq k$ . After constructing  $\mathbf{w}_{I_i}(s)$  for each  $i = 1, \dots, k$ , we define a smooth curve  $\mathbf{w}(s) = (w_1(s), \dots, w_n(s))$  by the summation

$$\begin{aligned}\mathbf{w}(s) &= \mathbf{w}_{J^c}(s) + \mathbf{w}_J(s), \\ \mathbf{w}_J(s) &= \mathbf{w}_{I_1}(s) + \dots + \mathbf{w}_{I_k}(s).\end{aligned}$$

Then  $\mathbf{w}(s)$  satisfies

$$\begin{aligned}f_{II,t}(\mathbf{w}(s)) &= f'(\mathbf{w}(s), \bar{\mathbf{w}}(s)) = \sum_{i=1}^k f_i(\mathbf{w}_{I_i}(s), \bar{\mathbf{w}}_{I_i}(s)) \\ &= (s+1) \sum_{i=1}^k f_i(\mathbf{w}_{I_i}, \bar{\mathbf{w}}_{I_i}) = (s+1)f'(\mathbf{w}, \bar{\mathbf{w}}) \\ &= (s+1)f_{II,t}(\mathbf{w}, \bar{\mathbf{w}}) \equiv 0, \\ \frac{d\|\mathbf{w}(s)\|^2}{ds}\big|_{s=0} &= \sum_{i=1}^k \frac{d\|\mathbf{w}_{I_i}(s)\|^2}{ds}\big|_{s=0} > 0.\end{aligned}$$

Thus defining  $\mathbf{v} := \frac{d\mathbf{w}}{ds}(0)$ , we conclude  $\mathbf{v} \in T_{\mathbf{w}}V_t \setminus T_{\mathbf{w}}S_{r_0}^{2n-1}$ . This completes the proof of the transversality.

**Remark 1.** In the above argument, if  $v_n$  is a vertex of the graph of  $f_i$  and it is not the right end vertex, we use the expression  $I_i = \{j \bmod n \mid \nu_i \leq j \leq \mu_i\}$  with  $\mu_i > n$ . This implies that

$$f_i(\mathbf{z}_{I_i}) = \sum_{j=\nu_i}^{\mu_i-1} z_j^{a_j} z_{j+1} \{|z_j|^{2b_j}(1-t) + t\}$$

where  $z_{j+n} = z_j, a_{j+n} = a_j, b_{j+n} = b_j$ . We do the same argument as above starting the right end variable  $z_{\mu_i} = z_{\mu_i-n}$ .

### 3.4. Applications.

**Corollary 1.** Let  $V_t$  be the hypersurface defined by  $f_{II,t}$  and let  $K_{t,r}$  be its link. Then there exists an isotopy  $\psi_t : (S_r^{2n-1}, K_{0,r}) \rightarrow (S_r^{2n-1}, K_{t,r})$  for  $0 \leq t \leq 1$  with  $\psi_0 = \text{id}$ .

This is immediate from Ehresmann's fibration theorem ([9]). As for the Milnor fibration of the second type, we have:

**Corollary 2.** For a fixed  $r > 0$ , there exists a positive real number  $\eta_0$  so that  $f_{II,t}^{-1}(\eta)$  and  $S_r^{2n-1}$  intersect transversely for any  $\eta$ ,  $|\eta| \leq \eta_0$  and  $0 \leq t \leq 1$ . In particular this implies that there exists a family of diffeomorphisms  $\psi_t : \partial E_0(\eta_0, r) \rightarrow \partial E_t(\eta_0, r)$  such that the following diagram is commutative:

$$\begin{array}{ccc}\partial E_0(\eta_0, r) & \xrightarrow{\psi_t} & \partial E_t(\eta_0, r) \\ \downarrow f_{II,0} & & \downarrow f_{II,t} \\ S_{\eta_0}^1 & = & S_{\eta_0}^1\end{array}$$

where  $\partial E_t(\eta_0, r) = \{\mathbf{z} \in \mathbb{C}^n \mid |f_{II,t}(\mathbf{z})| = \eta_0, \|\mathbf{z}\| \leq r\}$ .

*Proof.* Fix a positive real number  $r$ . Let

$$\begin{aligned}\partial \mathcal{E}(\eta_0, r) &:= \{(\mathbf{z}, t) \in \mathbb{C}^n \times [0, 1] \mid |f_{II,t}(\mathbf{z})| = \eta_0, \|\mathbf{z}\| \leq r\} \\ \partial^2 \mathcal{E}(\eta_0, r) &:= \{(\mathbf{z}, t) \in \mathbb{C}^n \times [0, 1] \mid |f_{II,t}(\mathbf{z})| = \eta_0, \|\mathbf{z}\| = r\}.\end{aligned}$$

Since  $S_r^{2n-1}$  intersects with  $V_t$  transversely and  $S_r^{2n-1} \cap V_t$  is compact for any  $0 \leq t \leq 1$ , there exists a positive real number  $\eta_0$  such that  $f_{II,t}^{-1}(\eta)$  and  $S_r^{2n-1}$  intersect transversely for any  $\eta, |\eta| \leq \eta_0$  and  $0 \leq t \leq 1$ . Thus the projection  $\pi' : (\partial \mathcal{E}(\eta_0, r), \partial^2 \mathcal{E}(\eta_0, r)) \rightarrow [0, 1]$  is a proper submersion. By the Ehresmann's fibration theorem [9],  $\pi'$  is a locally trivial fibration over  $[0, 1]$ . So the projection  $\pi'$  induces a family of isomorphisms  $\psi_t : \partial E_0(\eta_0, r) \rightarrow \partial E_t(\eta_0, r)$  of fibrations for any  $\mathbf{z}$  with  $|f_{II,t}(\mathbf{z})| \leq \eta_0$  and  $0 \leq t \leq 1$ .  $\square$

Now we consider again Milnor fibration of the link complement. Consider the mapping

$$(8) \quad f_{II,t}/|f_{II,t}| : S_r^{2n-1} \setminus K_{t,r} \rightarrow S^1.$$

As  $f_{II,t}(\mathbf{z}, \bar{\mathbf{z}})$  is polar weighted homogeneous polynomial, the  $S^1$ -action gives non-vanishing vector field, denoted as  $\frac{\partial}{\partial \theta}$  on  $S_r^{2n-1} \setminus K_{t,r}$  so that  $f_{II,t}(s \circ \mathbf{z}) = s^{d_p} f_{II,t}(\mathbf{z})$  for  $s \in S^1$ , this gives fibration structure for (8) for any  $r > 0$  and we call it a *spherical Milnor fibration* or a *Milnor fibration of the first description*. The isomorphism class of the fibration does not depend on  $r$ . Consider two fibrations

$$f_{II,t} : \partial E_t(\eta_0, r) \rightarrow S_{\eta_0}^1, \quad f_{II,t}/|f_{II,t}| : S_r^{2n-1} \setminus K_{t,r} \rightarrow S^1.$$

The first fibration is called a *Milnor fibration of the second description* or a *tubular Milnor fibration*. The isomorphism class of the tubular fibration does not depend on the choice of  $r$  and  $\eta_0 \ll r$ . As we know that two fibrations are isomorphic for sufficiently small  $r > 0$  and any  $t$  ([3, Theorem 36]), they are isomorphic for any  $r$ . Combining this and Corollary 2, we can sharpen Corollary 1 as follows.

**Corollary 3.** *Let  $\psi_t : (S_r^{2n-1}, K_{0,r}) \rightarrow (S_r^{2n-1}, K_{t,r})$  be an isotopy in Corollary 1.  $\psi_t$  can be constructed so that the following diagram is commutative.*

$$\begin{array}{ccc} S_r^{2n-1} \setminus K_{0,r} & \xrightarrow{\psi_t} & S_r^{2n-1} \setminus K_{t,r} \\ \downarrow f_{II,0}/|f_{II,0}| & & \downarrow f_{II,t}/|f_{II,t}| \\ S^1 & \xrightarrow{id} & S^1 \end{array}$$

Taking  $t = 1$ , we get a positive answer to the conjecture in [4].

*Proof.* Choose a positive real number  $\eta_0$  as in Corollary 2. Consider the cobordism variety  $\mathcal{V}_r := \{(\mathbf{z}, t) \in S_r^{2n-1} \times [0, 1] \mid f_{II,t}(\mathbf{z}, \bar{\mathbf{z}}) = 0\}$  and its open neighborhood  $\mathcal{W}_\eta := \{(\mathbf{z}, t) \in S_r^{2n-1} \times [0, 1] \mid |f_{II,t}(\mathbf{z})| < \eta\}$  of  $\mathcal{V}_t$ . Consider the projection mapping

$$\pi : S_r^{2n-1} \times [0, 1] \rightarrow [0, 1], \quad (\mathbf{z}, t) \mapsto t.$$

Let  $\frac{\partial}{\partial \theta}'$  be the projection of the gradient vector of  $\Im \log f_{II,t}(\mathbf{z}, \bar{\mathbf{z}})$  to the tangent space of  $S_r^{2n-1} \times [0, 1] \setminus \mathcal{V}_t$ . Using the vector field  $\frac{\partial}{\partial \theta}$  on  $S_r^{2n-1} \times [0, 1]$ , we see easily that  $\frac{\partial}{\partial \theta}'$  is a non-vanishing vector on  $S_r^{2n-1} \times [0, 1] \setminus \mathcal{V}_t$  which is linearly independent with  $\frac{\partial}{\partial t}$  over  $\mathbb{R}$ . Now we construct a vector field  $\mathcal{X}$  on  $S_r^{2n-1} \times [0, 1] \setminus \mathcal{V}_t$  such that

- (1)  $d\pi_*(\mathcal{X}(\mathbf{z}, t)) = \frac{\partial}{\partial t}$  and  $\{\mathcal{X}(\mathbf{z}, t), \frac{\partial}{\partial \theta}'(\mathbf{z}, t)\}$  are orthogonal.
- (2) For  $(\mathbf{z}, t) \in \mathcal{W}_{\eta_0/2}$ ,  $\{\mathcal{X}(\mathbf{z}, t), \text{grad}|f_{II,t}|(\mathbf{z}, t)\}$  are also orthogonal.

The condition (1) implies the argument of  $f_{II,t}$  does not change along the integral curve of  $\mathcal{X}$ . The conditions (1) and (2) implies the integral curve of  $\mathcal{X}$  keeps the level  $f_{II,t} = \eta$  for any  $\eta$  with  $|\eta| \leq \eta_0/2$ . Thus integral curves of vector field  $\mathcal{X}$  exists over  $[0, 1]$  and we construct the isotopy  $\psi_t$  using the integration curves of  $\mathcal{X}$ .  $\square$

**Remark 2.** Let  $f(\mathbf{z}, \bar{\mathbf{z}}) = \sum_{i=1}^m c_i \mathbf{z}^{\nu_i} \bar{\mathbf{z}}^{\mu_i}$  be a full simplicial mixed polynomial and  $g(\mathbf{z})$  be the associated Laurent polynomial of  $f$ . The last author defined a canonical diffeomorphism of  $\varphi : \mathbb{C}^{*n} \rightarrow \mathbb{C}^{*n}$  as follows ([2]):

$$\varphi : \mathbb{C}^{*n} \rightarrow \mathbb{C}^{*n},$$

$$\mathbf{z} = (\rho_1 \exp(i\theta_1), \dots, \rho_n \exp(i\theta_n)) \mapsto \mathbf{w} = (\xi_1 \exp(i\theta_1), \dots, \xi_n \exp(i\theta_n))$$

where  $(\rho_1, \dots, \rho_n)$  and  $(\xi_1, \dots, \xi_n)$  satisfy

$$(N + M) \begin{pmatrix} \log \rho_1 \\ \vdots \\ \log \rho_n \end{pmatrix} = (N - M) \begin{pmatrix} \log \xi_1 \\ \vdots \\ \log \xi_n \end{pmatrix}$$

where  $N = (\nu_1, \dots, \nu_n)$  and  $M = (\mu_1, \dots, \mu_n)$ . Then  $\varphi$  satisfies that  $\varphi(\mathbb{C}^{*n} \cap f^{-1}(c)) = \mathbb{C}^{*n} \cap g^{-1}(c)$  for any  $c \in \mathbb{C}$  ([2, Theorem 10]). However  $\varphi$  cannot be extended to a homeomorphism of  $\mathbb{C}^n \setminus \{O\}$  to itself in general, except the case of mixed Brieskorn polynomial.

**Example 1.** We will give an example of the above remark. Let  $f(\mathbf{z}, \bar{\mathbf{z}})$  be a simplicial polynomial defined by

$$f(\mathbf{z}, \bar{\mathbf{z}}) = z_1^3 \bar{z}_1 z_2 + z_2^3 \bar{z}_2 z_3 + z_3^3 \bar{z}_3 z_1.$$

Then the diffeomorphism of  $\varphi : \mathbb{C}^{*3} \rightarrow \mathbb{C}^{*3}$ ,  $\mathbf{z} = (z_1, z_2, z_3) \mapsto \mathbf{w} = (w_1, w_2, w_3)$  is given by

$$\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} |z_1|^{\frac{17}{9}} |z_2|^{\frac{-4}{9}} |z_3|^{\frac{2}{9}} \exp(i\theta_1) \\ |z_1|^{\frac{2}{9}} |z_2|^{\frac{17}{9}} |z_3|^{\frac{-4}{9}} \exp(i\theta_2) \\ |z_1|^{\frac{-4}{9}} |z_2|^{\frac{2}{9}} |z_3|^{\frac{17}{9}} \exp(i\theta_3) \end{pmatrix}.$$

The above map cannot extend to a continuous map on the coordinate planes  $\{(z_1, z_2, z_3) \in \mathbb{C}^3 \mid z_1 z_2 z_3 = 0\}$  as the negative exponents in the above description. So the map  $\varphi$  cannot extend to a homeomorphism of  $\mathbb{C}^3 \setminus \{O\}$  to itself.

## REFERENCES

- [1] J. L. Cisneros-Molina, *Join theorem for polar weighted homogeneous singularities*, Singularities II, edited by J. P. Brasselet, J. L. Cisneros-Molina, D. Massey, J. Seade and B. Teissier, Contemp. Math. **475**, Amer. Math. Soc., Providence, RI, 2008, 43–59.
- [2] M. Oka, *Topology of polar weighted homogeneous hypersurfaces*, Kodai Math. J. **31** (2008), 163–182.
- [3] M. Oka, *Non-degenerate mixed functions*, Kodai Math. J. **33** (2010), 1–62.
- [4] M. Oka, *On Mixed Brieskorn variety*, Contemp. Math. **538** (2011), 389–399.
- [5] P. Orlik and P. Wagreich, *Isolated singularities of algebraic surfaces with  $\mathbb{C}^*$  action*, Ann. of Math. **93** (1971), 205–228.
- [6] M. A. S. Ruas, J. Seade and A. Verjovsky, *On real singularities with a Milnor fibration*, Trends Math., edited by A. Libgober and M. Tibăr, Birkhäuser, Basel, 2003, 191–213.
- [7] J. Seade, *Fibered links and a construction of real singularities via complex geometry*, Bull. Braz. Math. Soc., **27** (1996), 199–215.
- [8] J. Seade, *On the Topology of Isolated Singularities in Analytic Spaces*, Progress in Mathematics vol. 241, Birkhäuser, 2005.
- [9] J. A. Wolf, *Differentiable fibre spaces and mappings compatible with Riemannian metrics*, Michigan Math. J. **11** (1964), 65–70.

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